Interpolating between the Bose-Einstein and the Fermi-Dirac distributions in odd dimensions

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Abstract

We consider the response of a uniformly accelerated monopole detector that is coupled to a *superposition* of an odd and an even power of a quantized, massless scalar field in flat spacetime in arbitrary dimensions. We show that, when the field is assumed to be in the Minkowski vacuum, the response of the detector is characterized by a Bose-Einstein factor in even spacetime dimensions, whereas a Bose-Einstein *as well as* a Fermi-Dirac factor appear in the detector response when the dimension of spacetime is odd. Moreover, we find that, it is possible to *interpolate* between the Bose-Einstein and the Fermi-Dirac distributions in odd spacetime dimensions by suitably adjusting the relative strengths of the detector's coupling to the odd and the even powers of the scalar field. We point out that the response of the detector is always thermal and we, finally, close by stressing the *apparent* nature of the appearance of the Fermi-Dirac factor in the detector response.

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I. INTRODUCTION

It is well-known that the response of a uniformly accelerated Unruh-DeWitt detector that is coupled to a quantized, massless scalar field is characterized by a Planckian distribution when the field is assumed to be in the Minkowski vacuum [1,2]. However, what does not seem to be so commonly known is the fact that this result is true only in even spacetime dimensions, and, in odd spacetime dimensions, a Fermi-Dirac factor (rather than a Bose-Einstein factor) appears in the response of the accelerated Unruh-DeWitt detector (for the original results, see Refs. [3–9]; for relatively recent discussions, see Refs. [10,11]).

The Unruh-DeWitt detector is a monopole detector that is coupled *linearly* to the quantum scalar field [1,2]. With a variety of motivations in mind, there has been a prevailing interest in literature in studying the response of detectors that are coupled *non-linearly* to the quantum field [12–16]. In a recent Letter [17], we had considered the response of a uniformly accelerated monopole detector that is coupled to an arbitrary (but, positive) integral power of a massless, quantum scalar field in $(D+1)$-dimensional flat spacetime. We had found that, when the detector is coupled to an even power of the scalar field, a Bose-Einstein factor arises in the response of the detector (in the Minkowski vacuum) in *all* spacetime dimensions, whereas, a Fermi-Dirac factor appears in the detector response *only* when *both* the spacetime dimension [viz. $(D+1)$] and the index of non-linearity of the coupling are odd.

In this note, we shall consider the response of an accelerated monopole detector that is coupled to a *superposition* of an odd and an even power of the massless, quantum scalar field. Though the response of such a detector in the Minkowski vacuum is characterized by the Bose-Einstein factor in even spacetime dimensions, interestingly, we find that, in odd spacetime dimensions, the response of the detector contains an admixture of the Bose-Einstein and the Fermi-Dirac factors. Also, as we shall see, it is possible to interpolate between the Bose-Einstein and the Fermi-Dirac factors in odd spacetime dimensions by suitably modulating the relative strengths of the detector's coupling to the odd and the even powers of the quantum scalar field. In what follows, we shall set $\hbar = c = k_{\rm B} = 1$ and, for convenience in notation, denote the trajectory $x^\mu(\tau)$ of the detector as $\tilde{x}(\tau)$ with τ being the proper time in the frame of the detector.

II. "INVERTED STATISTICS" FOR ODD COUPLINGS

Let us begin by reviewing our earlier result for a monopole detector that is coupled to the nth power (with n being a positive integer) of a real scalar field Φ through the following interaction Lagrangian [15,17]:

$$\mathcal{L}_{\rm NL} = \bar{c}\, m(\tau)\ \Phi^n\left[\tilde{x}(\tau)\right], \tag{1}$$

where $\bar{c}$ is a small coupling constant and $m(\tau)$ is the detector's monopole moment. Consider a situation wherein the quantum field $\hat{\Phi}$ is initially in the vacuum state $|0\rangle$ and the detector is in its ground state $|E_0\rangle$ corresponding to an energy eigen value E_0. Then, up to the first order in perturbation theory, the amplitude of transition of the detector to an excited state $|E\rangle$, corresponding to an energy eigen value E $(> E_0)$, is described by the integral [15,17]

$$\mathcal{A}_n(\mathcal{E}) = (i\bar{c}\mathcal{M}) \int\limits_{-\infty}^{\infty} d\tau\, e^{i\mathcal{E}\tau}\ \langle\Psi| : \hat{\Phi}^n[\tilde{x}(\tau)] : |0\rangle\,, \tag{2}$$

where $\mathcal{M} \equiv \langle E|\,\hat{m}(0)\,|E_0\rangle$, $\mathcal{E} = (E - E_0) > 0$, $|\Psi\rangle$ is the state of the quantum scalar field after its interaction with the detector and the colons denote normal ordering with respect to the Minkowski vacuum. (The normal ordering procedure is required to overcome the divergences that would otherwise arise for $n > 1$. For a detailed discussion on this point, see Refs. [15,17].) The transition probability of the detector to all possible final states $|\Psi\rangle$ of the quantum field is then given by

$$\mathcal{P}_n(\mathcal{E}) = \sum_{|\Psi\rangle} |\mathcal{A}_n(\mathcal{E})|^2 = \int\limits_{-\infty}^{\infty} d\tau \int\limits_{-\infty}^{\infty} d\tau'\, e^{-i\mathcal{E}(\tau-\tau')}\, G^{(n)}\left[\tilde{x}(\tau), \tilde{x}(\tau')\right], \tag{3}$$

where we have dropped an (irrelevant) overall factor of $(|\bar{c}||\mathcal{M}|)^2$ and $G^{(n)}\left[\tilde{x}(\tau), \tilde{x}(\tau')\right]$ is a $(2n)$-point function defined as

$$G^{(n)}\left[\tilde{x}(\tau), \tilde{x}(\tau')\right] = \langle 0| : \hat{\Phi}^n\left[\tilde{x}(\tau)\right] :: \hat{\Phi}^n\left[\tilde{x}(\tau')\right] : |0\rangle\,. \tag{4}$$

For trajectories wherein the $(2n)$-point function $G^{(n)}(\tau, \tau')$ $(\equiv G^{(n)}\left[\tilde{x}(\tau), \tilde{x}(\tau')\right])$ is invariant under translations in the proper time in the frame of the detector, as in the case of the Unruh-DeWitt detector, we can define a transition probability rate for the non-linearly coupled detector as follows:

$$\mathcal{R}_n(\mathcal{E}) = \int\limits_{-\infty}^{\infty} d\bar{\tau}\ e^{-i\mathcal{E}\bar{\tau}}\ G^{(n)}(\bar{\tau}), \tag{5}$$

where $\bar{\tau} = (\tau - \tau')$.

If we assume that the quantum field is in the Minkowski vacuum, then, using Wick's theorem, it is easy to show that, the $(2n)$-point function $G^{(n)}(\tilde{x}, \tilde{x}')$ above simplifies to [17]

$$G_{\mathrm{M}}^{(n)}(\tilde{x}, \tilde{x}') = (n!)\ \left[G_{\mathrm{M}}^{+}(\tilde{x}, \tilde{x}')\right]^n, \tag{6}$$

where $G_{\mathrm{M}}^{+}(\tilde{x}, \tilde{x}')$ denotes the Wightman function in the Minkowski vacuum. Along the trajectory of a detector that is accelerating uniformly with a proper acceleration g in a particular direction, the Wightman function for a massless scalar field in the Minkowski vacuum in $(D+1)$ spacetime dimensions (for $(D+1) \geq 3$) is given by [5,8,9]

$$G_{\mathrm{M}}^{+}(\bar{\tau}) = \left[\mathcal{C}_D\ (g/2i)^{(D-1)}\right]\ \Big(\sinh\left[(g\bar{\tau}/2) - i\epsilon\right]\Big)^{-(D-1)}, \tag{7}$$

where $\epsilon \to 0^+$ and $\mathcal{C}_D = \left[\Gamma\left[(D-1)/2\right] / \left(4\pi^{(D+1)/2}\right)\right]$ with $\Gamma\left[(D-1)/2\right]$ denoting the Gamma function. Therefore, along the trajectory of the uniformly accelerated detector, the $(2n)$-point function in the Minkowski vacuum (6) reduces to

$$G_{\mathrm{M}}^{(n)}(\bar{\tau}) = (n!)\ \left[\mathcal{C}_D^n\ (g/2i)^{\alpha}\right]\ \Big(\sinh\left[(g\bar{\tau}/2) - i\epsilon\right]\Big)^{-\alpha}, \tag{8}$$

where $\alpha = [(D-1)n]$. On substituting this $(2n)$-point function in the expression (5) and carrying out the integral, we find that the transition probability rate of the uniformly accelerated, non-linearly coupled detector can be expressed as [17]

$$\mathcal{R}_n(\mathcal{E}) = \mathcal{B}(n,D) \begin{cases} (g^\alpha/\mathcal{E}) \; \underbrace{[\exp(2\pi\mathcal{E}/g) - 1]^{-1}}_{\text{Bose-Einstein factor}} \prod\limits_{l=0}^{(\alpha-2)/2} [l^2 + (\mathcal{E}/g)^2] & \text{when } \alpha \text{ is even} \\ g^{(\alpha-1)} \; \underbrace{[\exp(2\pi\mathcal{E}/g) + 1]^{-1}}_{\text{Fermi-Dirac factor}} \prod\limits_{l=0}^{(\alpha-3)/2} [((2l+1)/2)^2 + (\mathcal{E}/g)^2] & \text{when } \alpha \text{ is odd,} \end{cases} \tag{9}$$

where the quantity $\mathcal{B}(n,D)$ is given by

$$\mathcal{B}(n,D) = (2\pi)\,(n!)\;\left[\mathcal{C}_D^n/\Gamma(\alpha)\right]. \tag{10}$$

Since, for even $(D+1)$, α is even for all n, a Bose-Einstein factor will always arise in the response of the uniformly accelerated detector in even-dimensional flat spacetimes. On the other hand, when $(D+1)$ is odd, clearly, α will be odd or even depending on whether n is odd or even. As a result, in odd-dimensional flat spacetimes, a Bose-Einstein factor will arise in the detector response only when n is even, but, as in the case of the Unruh-DeWitt detector, a Fermi-Dirac factor will appear when n is odd. [Note that the temperature associated with the Bose-Einstein and the Fermi-Dirac factors is the standard Unruh temperature, viz. $(g/2\pi)$.] Also, the response of the detector will be characterized completely by either a Bose-Einstein or a Fermi-Dirac distribution only in cases such that $\alpha < 3$ and, in situations wherein $\alpha \geq 3$, the detector response will contain, in addition to a Bose-Einstein or a Fermi-Dirac factor, a term which is polynomial in $(\mathcal{E}/g)$.

III. "MIXING STATISTICS" WITH A SUPERPOSITION OF ODD AND EVEN COUPLINGS

Now, consider a detector that interacts with the scalar field Φ through the following Lagrangian:

$$\mathcal{L}_{\rm SP} = m(\tau)\,\Big(\bar{c}_{\rm o}\,\Phi^{n_{\rm o}}\,[\tilde{x}(\tau)] + \bar{c}_{\rm e}\,\Phi^{n_{\rm e}}\,[\tilde{x}(\tau)]\Big), \tag{11}$$

where $\bar{c}_{\rm o}$ and $\bar{c}_{\rm e}$ denote two (small) coupling constants and $n_{\rm o}$ and $n_{\rm e}$ are positive integers. The transition amplitude of such a detector [under the same conditions as in the case of $\mathcal{A}_{\rm n}(\mathcal{E})$] up to the first order in the coupling constants $\bar{c}_{\rm o}$ and $\bar{c}_{\rm e}$ will be given by

$$\mathcal{A}_{\rm SP}(\mathcal{E}) = (i\mathcal{M}) \int\limits_{-\infty}^{\infty} d\tau\, e^{i\mathcal{E}\tau}\, \Big(\bar{c}_{\rm o}\,\langle\Psi|\,:\hat{\Phi}^{n_{\rm o}}[\tilde{x}(\tau)]:\,|0\rangle + \bar{c}_{\rm e}\,\langle\Psi|\,:\hat{\Phi}^{n_{\rm e}}[\tilde{x}(\tau)]:\,|0\rangle\Big), \tag{12}$$

where, as before, we have normal-ordered the matrix-elements in order to avoid the divergences. If we now assume that $n_{\rm o}$ is odd and $n_{\rm e}$ is even, then, being the expectation values of an odd power [viz. $(n_{\rm o}+n_{\rm e})$] of the quantum field, the cross terms in the corresponding transition probability vanish. As a result, we obtain that

$$\mathcal{P}_{\rm SP}(\mathcal{E}) = \int\limits_{-\infty}^{\infty} d\tau \int\limits_{-\infty}^{\infty} d\tau'\, e^{-i\mathcal{E}(\tau-\tau')} \left(G^{(n_{\rm e})}\left[\tilde{x}(\tau), \tilde{x}(\tau')\right] + r^2\, G^{(n_{\rm o})}\left[\tilde{x}(\tau), \tilde{x}(\tau')\right]\right), \tag{13}$$

where $G^{(n_{\rm e})}\left[\tilde{x}(\tau), \tilde{x}(\tau')\right]$ and $G^{(n_{\rm o})}\left[\tilde{x}(\tau), \tilde{x}(\tau')\right]$ are the $(2n)$-point functions [as defined in Eq. (4)] corresponding to $n_{\rm e}$ and $n_{\rm o}$ and the quantity $r = (|\bar{c}_{\rm o}|/|\bar{c}_{\rm e}|)$ denotes the relative strength of the detector's coupling to the odd power of the scalar field with respect to the even power. [As we had done earlier, in the above expression for $\mathcal{P}_{\rm SP}(\mathcal{E})$, we have dropped an unimportant overall factor of $(|\bar{c}_{\rm e}||\mathcal{M}|)^2$.] In situations wherein the $(2n)$-point functions are invariant under translations in the detector's proper time, the transition probability rate of the detector can be expressed as

$$\mathcal{R}_{\rm SP}(\mathcal{E}) = \left[\mathcal{R}_{n_{\rm e}}(\mathcal{E}) + r^2\, \mathcal{R}_{n_{\rm o}}(\mathcal{E})\right], \tag{14}$$

where $\mathcal{R}_n(\mathcal{E})$ denotes the transition probability rate defined in Eq. (5).

Therefore, for a detector that is in motion along a uniformly accelerated trajectory, when the field is assumed to be in the Minkowski vacuum, the quantities $\mathcal{R}_{n_{\rm e}}(\mathcal{E})$ and $\mathcal{R}_{n_{\rm o}}(\mathcal{E})$ in the expression for $\mathcal{R}_{\rm SP}(\mathcal{E})$ above will be given by $\mathcal{R}_n(\mathcal{E})$ in Eq. (9) corresponding to the even and the odd integers $n_{\rm e}$ and $n_{\rm o}$, respectively. Evidently, in such a case, in even spacetime dimensions, both $\mathcal{R}_{n_{\rm e}}(\mathcal{E})$ and $\mathcal{R}_{n_{\rm o}}(\mathcal{E})$ will be characterized by a Bose-Einstein factor. Whereas, in odd spacetime dimensions, $\mathcal{R}_{\rm SP}(\mathcal{E})$ will contain an admixture of the two distributions, with $\mathcal{R}_{n_{\rm e}}(\mathcal{E})$ being characterized by a Bose-Einstein factor, while $\mathcal{R}_{n_{\rm o}}(\mathcal{E})$ contains a Fermi-Dirac factor. Moreover, in odd spacetime dimensions, the detector response function $\mathcal{R}_{\rm SP}(\mathcal{E})$ can be *interpolated* between the Bose-Einstein and the Fermi-Dirac distributions by varying the quantity r (viz. the relative strength of the coupling constant $\bar{c}_{\rm o}$ with respect to $\bar{c}_{\rm e}$) from zero to infinity.

IV. DISCUSSION

An important point needs to be stressed regarding the appearance of the Fermi-Dirac factor in the response of a detector that is coupled to a scalar field. According to principle of detailed balance, a spectrum $\mathcal{R}_\beta(\mathcal{E})$ can be considered to be a thermal distribution at the inverse temperature β if the spectrum satisfies the following condition (see, for e.g., Refs. [7,9]):

$$\mathcal{R}_\beta(\mathcal{E}) = \left[e^{-\beta\mathcal{E}}\, \mathcal{R}_\beta(-\mathcal{E})\right]. \tag{15}$$

It is straightforward to check that this condition is always satisfied by the detector response functions $\mathcal{R}_n(\mathcal{E})$ and $\mathcal{R}_{\rm SP}(\mathcal{E})$ along the uniformly accelerated trajectory. Clearly, in spite of the appearance of polynomial terms as well as an admixture of the Bose-Einstein and the Fermi-Dirac factors, the response of the detectors is indeed thermal.

Actually, the principle of detailed balance is a consequence of the Kubo-Martin-Schwinger (KMS) condition according to which the Wightman function of a Bosonic field in thermal equilibrium at the inverse temperature β should be skew-periodic in imaginary proper time with a period β. (The Wightman function of a Fermionic field would be skew *and* anti-periodic in such a situation. For a discussion on this point, see, for instance, Refs. [7,9]). It

is straightforward to check that, along the uniformly accelerated trajectory, the Wightman function in the Minkowski vacuum (7) indeed satisfies the KMS condition (corresponding to the Unruh temperature) as required for a *Bosonic* field in *all* dimensions. If the Wightman function satisfies the KMS condition of a scalar field, then, obviously, all $(2n)$-point functions as well as a linear superposition of such functions that are constructed out of the Wightman function will also satisfy the same KMS condition. This immediately suggests that the "inversion of statistics", i.e. the appearance of a Fermi-Dirac factor in the response of a detector that is coupled to a scalar field, is an *apparent* phenomenon—it reflects a curious aspect of these detectors rather than point to any fundamental change in statistics in odd spacetime dimensions (it is for this reason that we have referred to statistics within quotes in the titles of the last two sections). Nevertheless, when models with (large and compact) extra dimensions are in vogue in literature, the fact that the characteristic response of an accelerated detector depends on the number of spacetime dimensions offers an interesting feature that can possibly be utilized to detect the extra dimensions [3]. These very reasons also provide sufficient motivation to examine whether the results presented in this note are generic to other spacetimes which exhibit real or accelerated horizons (such as, for e.g., the black hole, de Sitter and the anti-de Sitter spacetimes) [18].

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